Symmetric Fock space and orthogonal symmetric polynomials associated with the Calogero model

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Abstract. Using a similarity transformation that maps the Calogero model into N decoupled quantum harmonic oscillators, we construct a set of mutually commuting conserved operators of the model and their simultaneous eigenfunctions. The simultaneous eigenfunction is a deformation of the symmetrized number state (bosonic state) and forms an orthogonal basis of the Hilbert (Fock) space of the model. This orthogonal basis is different from the known one that is a variant of the Jack polynomial, i.e., the Hi-Jack polynomial. This fact shows that the conserved operators derived by the similarity transformation and those derived by the Dunkl operator formulation do not commute. Thus we conclude that the Calogero model has two, algebraically inequivalent sets of mutually commuting conserved operators, as is the case with the hydrogen atom. We also confirm the same story for the B_N-Calogero model.

1. Introduction

Exact solutions for the Schrödinger equations have provided interesting problems in physics and mathematical physics since the early days of quantum mechanics. Special functions such as the Hermite polynomials and the Laguerre polynomials play important roles in the study of the quantum harmonic oscillator and the hydrogen atom. Such a traditional approach to the quantum mechanics is enjoying a revived interest since the importance of the Jack symmetric polynomials was realized in the calculation of correlation functions of the Sutherland model [1–9]. The Sutherland model is one of the typical models among the one-dimensional quantum integrable systems with inverse-square long-range interactions that has been extensively studied with the helps of the Dunkl operator formulation and the theory of the Jack polynomials. Even an exact calculation of its dynamical density-density correlation function [8, 9] and an algebraic construction of its orthogonal basis [10, 11], i.e., the Jack polynomials, have been achieved.

The Calogero model is another typical model among the class [12]. Its Hamiltonian is given by

$$\hat{H}_{\mathrm{C}}^{(A)} = \frac{1}{2}\sum_{j=1}^{N}(p_j^2 + \omega^2 x_j^2) + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{a(a-1)}{(x_j - x_k)^2}, \tag{1.1}$$

where the constants a and ω are the coupling parameter and the strength of the external harmonic well, respectively, and p_j is a momentum operator, $p_j = -\mathrm{i}\frac{\partial}{\partial x_j}$. We note that the coordinate representation, or identification of the momentum with the partial differential operator, is implicitly assumed throughout the paper. The mass of the particles and the Planck constant $\hbar$ are taken to be unity. Strictly speaking the model (1.1) is introduced by Sutherland [13]. Calogero originally introduced the model

with harmonic interactions, which is obtained from the model with the harmonic well by fixing the center of mass at the coordinate origin as is presented in (1.1). The superscript (A) on the Hamiltonian means that it is invariant under the action of the A_{N-1}-type Weyl group, i.e. under S_N, on the indices of the particle. Thus the model is sometimes called the A_{N-1}-Calogero model [14]. Because of its structural similarity to the quantum harmonic oscillator, several ways of algebraic construction of the eigenfunctions of the Calogero Hamiltonian had been demonstrated [15–19] before the Rodrigues formula for the Jack polynomials appeared. However, identification of its orthogonal basis was missing for a long time [20]. Motivated by the Rodrigues formula for the Jack polynomials, we derived the Rodrigues formula for the Hi-Jack (or multivariable Hermite) polynomials for the first time [21, 22], which is now identified with an orthogonal basis for the Calogero model [23–25]. These studies mentioned above have stimulated lots of works on variants of Jack polynomials and integrable systems with inverse-square interactions. The multivariable Laguerre polynomials associated with the B_N-type Calogero model,

$$\hat{H}_{\mathrm{C}}^{(B)} = \frac{1}{2}\sum_{j=1}^{N}\left(p_j^2 + \omega^2 x_j^2 + \frac{b(b-1)}{x_j^2}\right) + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\left(\frac{a(a-1)}{(x_j-x_k)^2} + \frac{a(a-1)}{(x_j+x_k)^2}\right), \tag{1.2}$$

where a constant b is another coupling parameter besides that of the A_{N-1} case, has attracted lots of interest as such a variant [24–26]. The superscript (B) on the Hamiltonian means that it is invariant under the action of the B_N-type Weyl group on the indices of the particle. Through the Dunkl operator formulations for the above three models [27–29], we realize that the Jack, Hi-Jack and multivariable Laguerre polynomials are the simultaneous eigenfunctions of the conserved operators of the corresponding models [23, 26]. See Appendix A for detail.

Quite recently, Gurappa and Panigrahi presented similarity transformations that map the Calogero and B_N-Calogero models into N decoupled quantum harmonic oscillators [30, 31]. Their transformation for the A_{N-1} case is, in a sense, equivalent to a transformation to the Euler operator, $\sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j}$, which had been shown by Sogo [32]. Reformulating their results, we noticed the connection of the number operator and the symmetrized number state (bosonic state) of the harmonic oscillators with sets of conserved operators and symmetric orthogonal bases of the A_{N-1}- and B_N-Calogero models [33]. The purpose of this paper is to present detailed properties of the orthogonal basis derived by the similarity transformation method.

The outline of the paper is as follows. In section 2, we present similarity transformations from the Calogero models to the decoupled harmonic oscillators. In section 3, we construct a set of conserved operators from the number operators and their simultaneous eigenfunctions that form orthogonal bases of the models. Some properties of the simultaneous eigenfunctions are presented. In section 4, we compare

the new orthogonal bases with the known orthogonal bases. And we show that the conserved operators constructed by the similarity transformation method and those constructed by the Dunkl operator formulation are algebraically inequivalent. In section 5, we summarize the results and discuss future problems. Appendix A presents a brief summary on the Dunkl operator formulation and variants of the Jack polynomial. Appendix B covers detailed discussions on the cancellation of essential singularities. Appendix C shows explicit forms of some of the new orthogonal basis.

2. Similarity transformation to harmonic oscillators

We show a series of similarity transformations that map the Hamiltonians of the Calogero models to that of the N decoupled quantum harmonic oscillators [30, 31]. The ground state wave function $\Psi_{\mathrm{g}}^{(A)}$ and the ground state energy $E_{\mathrm{g}}^{(A)}$ for the A_{N-1}-Calogero model are given by

$$\Psi_{\mathrm{g}}^{(A)}(\boldsymbol{x}) = \prod_{1\leq j<k\leq N} |x_j - x_k|^a \ \exp\left(-\frac{1}{2}\omega\sum_{l=1}^{N} x_l^2\right), \tag{2.1}$$

$$E_{\mathrm{g}}^{(A)} = \frac{1}{2}\omega N((N-1)a+1). \tag{2.2}$$

An excited state is written by a product of the ground state and some symmetric polynomial $\phi(\boldsymbol{x})$, $\Psi^{(A)} = \phi(\boldsymbol{x})\Psi_{\mathrm{g}}^{(A)}$. Since we are interested in the symmetric polynomial part of the excited state, we perform a similarity transformation and remove the ground state from the operand of the Hamiltonian (1.1),

$$\begin{aligned} H_{\mathrm{C}}^{(A)} &\stackrel{\mathrm{def}}{=} (\Psi_{\mathrm{g}}^{(A)})^{-1}(\hat{H}_{\mathrm{C}}^{(A)} - E_{\mathrm{g}}^{(A)})\Psi_{\mathrm{g}}^{(A)} \\ &= \sum_{l=1}^{N}\left(-\frac{1}{2}\frac{\partial^2}{\partial x_l^2} + \omega x_l \frac{\partial}{\partial x_l}\right) - \frac{1}{2}a\sum_{\substack{l,m=1\\ l\neq m}}^{N}\frac{1}{x_l - x_m}\left(\frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_m}\right). \end{aligned} \tag{2.3}$$

We apply a similar procedure to the B_N-Calogero model (1.2). Using the ground state wave function $\Psi_{\mathrm{g}}^{(B)}$ and the ground state energy $E_{\mathrm{g}}^{(B)}$ for the B_N-Calogero model,

$$\Psi_{\mathrm{g}}^{(B)}(\boldsymbol{x}) = \prod_{1\leq j<k\leq N} |x_j - x_k|^a |x_j + x_k|^a \prod_{l=1}^{N} |x_l|^b \exp\left(-\frac{1}{2}\omega\sum_{m=1}^{N} x_m^2\right), \tag{2.4}$$

$$E_{\mathrm{g}}^{(B)} = \frac{1}{2}\omega N(2(N-1)a + 2b + 1), \tag{2.5}$$

we transform the B_N-Calogero model as

$$\begin{aligned} H_{\mathrm{C}}^{(B)} &\stackrel{\mathrm{def}}{=} (\Psi_{\mathrm{g}}^{(B)})^{-1}(\hat{H}_{\mathrm{C}}^{(B)} - E_{\mathrm{g}}^{(B)})\Psi_{\mathrm{g}}^{(B)} \\ &= \sum_{l=1}^{N}\left(-\frac{1}{2}\frac{\partial^2}{\partial x_l^2} - \frac{b}{x_l}\frac{\partial}{\partial x_l} + \omega x_l \frac{\partial}{\partial x_l}\right) - a\sum_{\substack{l,m=1\\ l\neq m}}^{N}\frac{1}{x_l^2 - x_m^2}\left(x_l\frac{\partial}{\partial x_l} - x_m\frac{\partial}{\partial x_m}\right). \end{aligned} \tag{2.6}$$

Though the discussions above seem to be restricted to the bosonic wave functions, they cover any choice of statistics. The statistics of the particles, or in other words, the symmetry of the wave functions of the Calogero model is determined by a choice of the phase of the Jastraw factor $\prod_{1\leq j<k\leq N}|x_j-x_k|^a$ in the ground state wave function. In fact, we can choose any phase. For instance, a function,

$$\prod_{1\leq j<k\leq N}|x_j-x_k|^a(\mathrm{sgn}(x_j-x_k))^m\exp\left(-\frac{1}{2}\omega\sum_{l=1}^{N}x_l^2\right), \tag{2.7}$$

where $0\leq m<2$, is the ground state of the A_{N-1}-Calogero model. This fact is related to impenetrability of the inverse square potential in one-dimension [12, 13]. There must be some deep physical meaning behind the fact, which is beyond the scope of this paper. We note that the B_N-Calogero model also has similar freedom in the choice of the phase of the ground state wave function. The choice of the phase has no effect in the following study. Thus we have taken the simplest choice as a representative. In what follows, we sometimes call the operators (2.3) and (2.6) Hamiltonians of the A_{N-1}- and B_N-Calogero models instead of the original Hamiltonians (1.1) and (1.2).

Introducing the Euler operator $\mathcal{O}_{\mathrm{E}}$ and the Lassalle operators $\mathcal{O}_{\mathrm{L}}^{(A,B)}$ [34, 35], we can rewrite the Hamiltonians of the A_{N-1}- and B_N-Calogero models in a unified fashion,

$$H_{\mathrm{C}}^{(A,B)}=\omega\mathcal{O}_{\mathrm{E}}-\frac{1}{2}\mathcal{O}_{\mathrm{L}}^{(A,B)}, \tag{2.8}$$

where

$$\mathcal{O}_{\mathrm{E}}\stackrel{\mathrm{def}}{=}\sum_{l=1}^{N}x_l\frac{\partial}{\partial x_l}, \tag{2.9a}$$

$$\mathcal{O}_{\mathrm{L}}^{(A)}\stackrel{\mathrm{def}}{=}\sum_{l=1}^{N}\frac{\partial^2}{\partial x_l^2}+a\sum_{\substack{l,m=1\\l\neq m}}^{N}\frac{1}{x_l-x_m}\left(\frac{\partial}{\partial x_l}-\frac{\partial}{\partial x_m}\right), \tag{2.9b}$$

$$\mathcal{O}_{\mathrm{L}}^{(B)}\stackrel{\mathrm{def}}{=}\sum_{l=1}^{N}\left(\frac{\partial^2}{\partial x_l^2}+\frac{2b}{x_l}\frac{\partial}{\partial x_l}\right)+2a\sum_{\substack{l,m=1\\l\neq m}}^{N}\frac{1}{x_l^2-x_m^2}\left(x_l\frac{\partial}{\partial x_l}-x_m\frac{\partial}{\partial x_m}\right). \tag{2.9c}$$

Since the commutation relations between the Euler operator $\mathcal{O}_{\mathrm{E}}$ and the Lassalle operators $\mathcal{O}_{\mathrm{L}}^{(A,B)}$ are

$$[\mathcal{O}_{\mathrm{L}}^{(A,B)},\mathcal{O}_{\mathrm{E}}]=2\mathcal{O}_{\mathrm{L}}^{(A,B)}, \tag{2.10}$$

both Hamiltonians have the same algebraic structure. For a while, we omit the superscript (A) and (B) to avoid complexity and duplication of the expressions. Through (2.10) and the Baker-Hausdorff formula, we confirm that the Hamiltonians are transformed into the Euler operator [32],

$$\mathrm{e}^{\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}}H_{\mathrm{C}}\mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}}=\omega\mathcal{O}_{\mathrm{E}}, \tag{2.11}$$

which gives decompositions of the two models into the total momentum operator for N interaction-free particles on a ring of circumference L with the identification $x_j = \exp\frac{2\pi\mathrm{i}}{L}\theta_j$.

Furthermore, we transform the Euler operator into the Hamiltonian of the decoupled quantum harmonic oscillators. The following commutation relations,

$$[\triangle, \mathcal{O}_{\mathrm{E}}] = 2\triangle, \quad [\boldsymbol{x}^2, \mathcal{O}_{\mathrm{E}}] = -2\boldsymbol{x}^2, \quad [\triangle, \boldsymbol{x}^2] = 2(2\mathcal{O}_{\mathrm{E}} + N), \tag{2.12}$$

where the symbols $\triangle$ and $\boldsymbol{x}^2$ denote the Laplacian and the square of the norm,

$$\triangle \stackrel{\mathrm{def}}{=} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}, \qquad \boldsymbol{x}^2 \stackrel{\mathrm{def}}{=} \sum_{j=1}^{N} x_j^2, \tag{2.13}$$

and again the Baker-Hausdorff formula yield

$$\mathrm{e}^{-\frac{1}{4\omega}\triangle}\omega\mathcal{O}_{\mathrm{E}}\mathrm{e}^{\frac{1}{4\omega}\triangle} = \omega\mathcal{O}_{\mathrm{E}} - \frac{1}{2}\triangle. \tag{2.14}$$

Finally, the similarity transformation using the Gaussian kernel produces the Hamiltonian of the N decoupled quantum harmonic oscillators with its ground state energy subtracted from it,

$$\mathrm{e}^{-\frac{1}{2}\omega\boldsymbol{x}^2}\mathrm{e}^{-\frac{1}{4\omega}\triangle}\omega\mathcal{O}_{\mathrm{E}}\mathrm{e}^{\frac{1}{4\omega}\triangle}\mathrm{e}^{\frac{1}{2}\omega\boldsymbol{x}^2} = \frac{1}{2}\sum_{j=1}^{N}(p_j^2 + \omega^2 x_j^2) - \frac{1}{2}N\omega. \tag{2.15}$$

In terms of the creation and annihilation operators of the quantum harmonic oscillators,

$$a_j^\dagger = \frac{1}{2\omega\mathrm{i}}(p_j + \mathrm{i}\omega x_j), \tag{2.16a}$$

$$a_j = \mathrm{i}(p_j - \mathrm{i}\omega x_j), \tag{2.16b}$$

$$n_j = a_j^\dagger a_j = \frac{1}{2\omega}(p_j^2 + \omega^2 x_j^2) - \frac{1}{2}, \tag{2.16c}$$

the r.h.s. of (2.15) becomes the sum of the number operators, $\omega\sum_{j=1}^{N} n_j$. To summarize, we get the similarity transformations,

$$T^{-1}H_{\mathrm{C}}T = \omega\sum_{j=1}^{N} n_j, \quad T \stackrel{\mathrm{def}}{=} \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}}\mathrm{e}^{\frac{1}{4\omega}\triangle}\mathrm{e}^{\frac{1}{2}\omega\boldsymbol{x}^2}, \tag{2.17}$$

which map the A_{N-1}- and B_N-Calogero Hamiltonians to that of the N decoupled quantum harmonic oscillators. The number operators, n_j, $j = 1, 2, \cdots, N$, are mutually commuting conserved operators of the quantum harmonic oscillators. Their non-degenerate simultaneous eigenfunctions are nothing but the (nonsymmetric) number states,

$$|n_1, \cdots, n_N\rangle \stackrel{\mathrm{def}}{=} \prod_{j=1}^{N}(a_j^\dagger)^{n_j}|0\rangle, \tag{2.18}$$

where $|0\rangle \stackrel{\text{def}}{=} \mathrm{e}^{-\frac{1}{2}\omega \boldsymbol{x}^2}$ is the vacuum state for the quantum harmonic oscillators. Note that we implicitly employ the coordinate representation, $|0\rangle \sim \langle \boldsymbol{x}|0\rangle$. We are tempted to conclude that the similarity transformation of the number state $T|n_1, \cdots, n_N\rangle$ back to the Hilbert space of the Calogero models gives the non-symmetric orthogonal basis of the models. However, as we shall see in the next section, this conclusion is wrong because of a specific property of the Lassalle operators.

3. Conserved operators and orthogonal bases

Let us consider the similarity transformation from the number state back to the Hilbert space of the Calogero models. It is easy to verify,

$$x_j = \mathrm{e}^{\frac{1}{4\omega}\triangle} \mathrm{e}^{\frac{1}{2}\omega \boldsymbol{x}^2} a_j^\dagger \mathrm{e}^{-\frac{1}{2}\omega \boldsymbol{x}^2} \mathrm{e}^{-\frac{1}{4\omega}\triangle}, \tag{3.1a}$$

$$\frac{\partial}{\partial x_j} = \mathrm{e}^{\frac{1}{4\omega}\triangle} \mathrm{e}^{\frac{1}{2}\omega \boldsymbol{x}^2} a_j \mathrm{e}^{-\frac{1}{2}\omega \boldsymbol{x}^2} \mathrm{e}^{-\frac{1}{4\omega}\triangle}, \tag{3.1b}$$

$$x_j \frac{\partial}{\partial x_j} = \mathrm{e}^{\frac{1}{4\omega}\triangle} \mathrm{e}^{\frac{1}{2}\omega \boldsymbol{x}^2} n_j \mathrm{e}^{-\frac{1}{2}\omega \boldsymbol{x}^2} \mathrm{e}^{-\frac{1}{4\omega}\triangle}. \tag{3.1c}$$

Then the similarity transformation of the number state is expressed by the monomial acted by the exponentiation of the Lassalle operators,

$$T|n_1, \cdots, n_N\rangle = \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}} x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}, \tag{3.2}$$

for both A_{N-1} and B_N cases. However, as we can see in Appendix B, acting the Lassalle operator infinitely many times on a monomial generates essential singularities at $x_i = x_j$ for both cases and in addition at $x_i = -x_j$ and $x_i = 0$ for the B_N case. Thus we have to consider some escape from such essential singularities in order to make physical eigenfunctions for the Calogero models.

The keys to such an escape are symmetrization for both cases and additional restriction to even parity for the B_N case. Here we introduce two symmetrized number states (bosonic states) which respectively correspond to the A_{N-1} and B_N cases as

$$|\lambda\rangle \stackrel{\text{def}}{=} \sum_{\substack{\sigma \in S_N \\ \text{distinct}}} |\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(N)}\rangle = m_\lambda(\boldsymbol{a}^\dagger)|0\rangle, \tag{3.3a}$$

$$|2\lambda\rangle \stackrel{\text{def}}{=} \sum_{\substack{\sigma \in S_N \\ \text{distinct}}} |2\lambda_{\sigma(1)}, \cdots, 2\lambda_{\sigma(N)}\rangle = m_\lambda((\boldsymbol{a}^\dagger)^2)|0\rangle = m_{2\lambda}(\boldsymbol{a}^\dagger)|0\rangle, \tag{3.3b}$$

where λ and m_λ are the Young diagram, or a partition of a nonnegative integer $|\lambda| \stackrel{\text{def}}{=} \sum_{j=1}^N \lambda_j$ of at most N parts, and the monomial symmetric function,

$$\lambda \stackrel{\text{def}}{=} \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\}, \tag{3.4}$$

$$
\begin{aligned}
&\lambda_k, \quad k=1,2,\cdots,N, \text{ are integers}, \\
&m_\lambda(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{\substack{\sigma\in S_N \\ \text{distinct}}} x_1^{\lambda_{\sigma(1)}} x_2^{\lambda_{\sigma(2)}} \cdots x_N^{\lambda_{\sigma(N)}}. \qquad (3.5)
\end{aligned}
$$

Note that the summation over distinct permutations is done so that each monomial appears only once. A simplified but ambiguous notation such as $m_\lambda(\boldsymbol{x}^2) \stackrel{\text{def}}{=} m_\lambda(x_1^2,\cdots,x_N^2)$ has been introduced to compactify arguments of multivariable functions. These symmetrized number states are the simultaneous eigenfunctions for any symmetrized functions, say, the power sums $P_l(n_1,\cdots,n_N)$, of the number operators,

$$P_l(\boldsymbol{n})|\lambda\rangle = P_l(\lambda)|\lambda\rangle, \tag{3.6a}$$

$$P_l(\boldsymbol{n})|2\lambda\rangle = P_l(2\lambda)|2\lambda\rangle, \tag{3.6b}$$

where

$$P_l(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{j=1}^{N} (x_j)^l. \tag{3.7}$$

We take the power sums of the number operators, $P_l(\boldsymbol{n})$, $l=1,2,\cdots,N$, as the set of commuting conserved operators of the harmonic oscillators. Consequently we regard the symmetrized number states (3.3) as the states that are uniquely identified by the set of quantum numbers,

$$\{P_1(\lambda),\cdots,P_N(\lambda)\}, \tag{3.8a}$$

$$\{P_1(2\lambda),\cdots,P_N(2\lambda)\}, \tag{3.8b}$$

for the A_{N-1} and B_N cases, respectively. Since they are eigenfunctions of Hermitian operators without degeneracy, they form orthogonal bases for the harmonic oscillators.

We define the dual bases for the states (3.3a) and (3.3b) by

$$\langle\lambda| \stackrel{\text{def}}{=} \langle 0|m_\lambda(\boldsymbol{a}), \tag{3.9a}$$

$$\langle 2\lambda| \stackrel{\text{def}}{=} \langle 0|m_{2\lambda}(\boldsymbol{a}), \tag{3.9b}$$

where the vacuum bra $\langle 0| = e^{-\frac{1}{2}\boldsymbol{x}^2}$. Rewriting the Young diagram λ as

$$\lambda = \{\underbrace{\lambda_1,\cdots,\lambda_1}_{n_1}, \underbrace{\lambda_{n_1+1},\cdots,\lambda_{n_1+1}}_{n_2}, \lambda_{n_1+n_2+1},\cdots\} = \{\lambda_1,\cdots,\lambda_N\}, \tag{3.10}$$

we confirm the orthogonality of the symmetrized number states,

$$\langle\mu|\lambda\rangle = \frac{N!}{n_1!n_2!\cdots}\prod_{j=1}^{N}\lambda_j!\,\langle 0|0\rangle\,\delta_{\lambda\mu}, \tag{3.11a}$$

$$\langle 2\mu|2\lambda\rangle = \frac{N!}{n_1!n_2!\cdots}\prod_{j=1}^{N}(2\lambda_j)!\,\langle 0|0\rangle\,\delta_{\lambda\mu}. \tag{3.11b}$$

We note that the dual bases are Hermitian conjugates of the symmetrized number states, which reflects the fact that the number operators of the harmonic oscillators are Hermitian operators.

By the transformation of symmetrized number states,

$$T^{(A)}|\lambda\rangle = e^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}^{(A)}} m_\lambda(\boldsymbol{x}) \stackrel{\mathrm{def}}{=} M_\lambda(\boldsymbol{x};1/a,\omega), \tag{3.12a}$$

$$T^{(B)}|2\lambda\rangle = e^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}^{(B)}} m_{2\lambda}(\boldsymbol{x}) \stackrel{\mathrm{def}}{=} Y_\lambda(\boldsymbol{x};1/a,1/b,\omega), \tag{3.12b}$$

we get the eigenfunctions of the Calogero models (2.3) and (2.6), or the symmetric polynomial parts of the eigenfunctions of the original Calogero models (1.1) and (1.2). They are, indeed, symmetric polynomials and do not have any essential singularities. Detailed discussions on the cancellation of essential singularities are presented in Appendix B.

Now we transform the symmetrized number state and get orthogonal bases for the Calogero models (1.1) and (1.2). We introduce the creation and annihilation operators for the Calogero model as

$$b_j^+ \stackrel{\mathrm{def}}{=} T a_j^\dagger T^{-1} = \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}} x_j \mathrm{e}^{\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}}, \tag{3.13a}$$

$$b_j \stackrel{\mathrm{def}}{=} T a_j T^{-1} = \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}} \frac{\partial}{\partial x_j} \mathrm{e}^{\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}}, \tag{3.13b}$$

$$\nu_j \stackrel{\mathrm{def}}{=} b_j^+ b_j. \tag{3.13c}$$

Including the action to the ground state wave function, we obtain the creation-annihilation operators for the original Calogero models (1.1) and (1.2),

$$\hat{b}_j^+ \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{g}} b_j^+ (\Psi_{\mathrm{g}})^{-1}, \tag{3.14a}$$

$$\hat{b}_j \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{g}} b_j (\Psi_{\mathrm{g}})^{-1}, \tag{3.14b}$$

$$\hat{\nu}_j \stackrel{\mathrm{def}}{=} \hat{b}_j^+ \hat{b}_j. \tag{3.14c}$$

In terms of the above creation operators, the eigenfunctions of the original Calogero models are

$$|\lambda\rangle^{(A)} \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{g}}^{(A)} T^{(A)}|\lambda\rangle = \Psi_{\mathrm{g}}^{(A)} \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}^{(A)}} m_\lambda(\boldsymbol{x}) = m_\lambda(\hat{\boldsymbol{b}}^{(A)+})|0\rangle^{(A)}, \tag{3.15a}$$

$$|\lambda\rangle^{(B)} \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{g}}^{(B)} T^{(B)}|2\lambda\rangle = \Psi_{\mathrm{g}}^{(B)} \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}^{(B)}} m_\lambda(\boldsymbol{x}^2) = m_\lambda((\hat{\boldsymbol{b}}^{(B)+})^2)|0\rangle^{(B)}, \tag{3.15b}$$

where $|0\rangle^{(A)} \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{g}}^{(A)}$ and $|0\rangle^{(B)} \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{g}}^{(B)}$ are the ground states for the original Calogero models. The eigenfunctions (3.15) simultaneously diagonalize all the mutually commuting conserved operators,

$$P_l(\hat{\boldsymbol{\nu}}) = \sum_{j=1}^{N} (\hat{\nu}_j)^l, \quad l = 1, 2, \cdots, N. \tag{3.16}$$

The dual bases are defined in a similar way to that of the quantum harmonic oscillators,

$$ {}^{(A)}\langle\lambda| \stackrel{\text{def}}{=} {}^{(A)}\langle 0|m_\lambda(\hat{\boldsymbol{b}}^{(A)}), \tag{3.17a} $$
$$ {}^{(B)}\langle\lambda| \stackrel{\text{def}}{=} {}^{(B)}\langle 0|m_\lambda((\hat{\boldsymbol{b}}^{(A)})^2), \tag{3.17b} $$

where ${}^{(A)}\langle 0| = \Psi_{\mathrm{g}}^{(A)}$ and ${}^{(B)}\langle 0| = \Psi_{\mathrm{g}}^{(B)}$. Their orthogonality is also confirmed in a similar way,

$$ {}^{(A)}\langle\mu|\lambda\rangle^{(A)} = \frac{N!}{n_1!n_2!\cdots}\prod_{j=1}^{N}\lambda_j!\,{}^{(A)}\langle 0|0\rangle^{(A)}\,\delta_{\lambda\mu}, \tag{3.18a} $$
$$ {}^{(B)}\langle\mu|\lambda\rangle^{(B)} = \frac{N!}{n_1!n_2!\cdots}\prod_{j=1}^{N}(2\lambda_j)!\,{}^{(B)}\langle 0|0\rangle^{(B)}\,\delta_{\lambda\mu}, \tag{3.18b} $$

and the vacuum normalizatin constants are

$$ \begin{aligned} {}^{(A)}\langle 0|0\rangle^{(A)} = {} & (\frac{1}{2\omega})^{\frac{N(Na+(1-a))}{2}}(2\pi)^{\frac{N}{2}}N! \\ & \times\prod_{1\le j<k\le N}\frac{\Gamma((k-j+1)a)\Gamma(1+(k-j+1)a)}{\Gamma((k-j)a)\Gamma(1+(k-j)a)} \\ & \times\prod_{1\le j\le N}\Gamma(1+(N-j)a), \end{aligned} \tag{3.19a} $$

$$ \begin{aligned} {}^{(B)}\langle 0|0\rangle^{(B)} = {} & (\frac{1}{\omega})^{N(N-1)a+N(b+\frac{1}{2})}N! \\ & \times\prod_{1\le j<k\le N}\frac{\Gamma((k-j+1)a)\Gamma(1+(k-j+1)a)}{\Gamma((k-j)a)\Gamma(1+(k-j)a)} \\ & \times\prod_{1\le j\le N}\Gamma(1+(N-j)a)\Gamma((N-j)a+b+\frac{1}{2}), \end{aligned} \tag{3.19b} $$

where $\Gamma(z)$ denotes the gamma functions. A proof of the vacuum normalization constants is given in [24, 25].

As is similar to the triangularity of the Hi-Jack polynomials [22], polynomial parts of these eigenfunctions possess the triangularity,

$$ M_\lambda(\boldsymbol{x};1/a,\omega) = m_\lambda(\boldsymbol{x}) + \sum_{\substack{\mu\stackrel{\mathrm{d}}{<}\lambda,\ |\mu|<|\lambda| \\ |\mu|\equiv|\lambda| \pmod 2}}\left(-\frac{1}{4\omega}\right)^{(|\lambda|-|\mu|)/2} w_{\lambda\mu}^{(A)}(a)m_\mu(\boldsymbol{x}), \tag{3.20a} $$

$$ Y_\lambda(\boldsymbol{x};1/a,1/b,\omega) = m_\lambda(\boldsymbol{x}^2) + \sum_{\mu\stackrel{\mathrm{d}}{<}\lambda,\ |\mu|<|\lambda|}\left(-\frac{1}{4\omega}\right)^{|\lambda|-|\mu|} w_{\lambda\mu}^{(B)}(a,b)m_\mu(\boldsymbol{x}^2), \tag{3.20b} $$

with respect to the weak dominance order,

$$ \mu\stackrel{\mathrm{d}}{<}\lambda \Leftrightarrow \mu\neq\lambda \text{ and } \sum_{k=1}^{l}\mu_k \le \sum_{k=1}^{l}\lambda_k \text{ for all } l=1,2,\cdots,N. \tag{3.21} $$

The coefficients $w^{(A)}_{\lambda\mu}(a)$ and $w^{(B)}_{\lambda\mu}(a,b)$ are polynomials of the coupling parameters with integer coefficients, which is similar to integrality of the Jack and Hi-Jack polynomials [4, 22]. Explicit forms of some of the above symmetric orthogonal polynomials are shown in Appendix C. We note that the above orthogonal symmetric polynomials, (3.15a) and (3.15b), can be interpreted as a multivariable generalization of the Hermite polynomial and a multivariable generalization of the Laguerre polynomial, respectively.

4. Relationships between new and known orthogonal bases

As we mentioned before, there are known orthogonal bases for the A_{N-1}- and B_N-Calogero models, namely, the Hi-Jack and the multivariable Laguerre polynomials that are variants of the Jack polynomials. We shall compare the new and the known orthogonal bases here.

We use the formulae that relate the Hi-Jack polynomial $j_\lambda(\boldsymbol{x};1/a,\omega)$ and the multivariable Laguerre polynomial $l_\lambda(\boldsymbol{x};1/a,1/b,\omega)$ with the Jack polynomial $J_\lambda(\boldsymbol{x};1/a)$ [24, 32, 34, 35],

$$j_\lambda(\boldsymbol{x};1/a,\omega)=J_\lambda(\boldsymbol{\alpha}^{(A)\dagger};1/a)\cdot 1=\mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}^{(A)}_{\mathrm{L}}}J_\lambda(\boldsymbol{x};1/a), \tag{4.1a}$$

$$l_\lambda(\boldsymbol{x};1/a,1/b,\omega)=J_\lambda((\boldsymbol{\alpha}^{(B)\dagger})^2;1/a)\cdot 1=\mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}^{(B)}_{\mathrm{L}}}J_\lambda(\boldsymbol{x}^2;1/a). \tag{4.1b}$$

The definitions of the Dunkl operators, $\alpha^{(A)\dagger}_k$ and $\alpha^{(B)\dagger}_k$, and the Jack polynomial $J_\lambda(\boldsymbol{x})$ are presented in Appendix A. Those variants of the Jack polynomial are respectively different from the new orthogonal bases (3.15) for the corresponding models obtained in the previous section and do not diagonalize the conserved operators (3.16). On the other hand, the Hi-Jack and the multivariable Laguerre polynomials are uniquely identified as the simultaneous eigenfunctions of corresponding sets of conserved operators, $I^{(A)}_k$ and $I^{(B)}_k$ for $k=1,2,\cdots,N$, given by the Dunkl operator formulation. This means that the new orthogonal bases are not the simultaneous eigenfunctions for these conserved operators.

For the sake of fairness , we should note that the "new" orthogonal bases are, in a sense, "old" because they are nothing but what was given by Brink, Hansson, Konstein and Vasiliev for the A_{N-1}-Calogero model [16, 17]. A proof is as follows. The Jack polynomials have triangular expansion in the monomial symmetric functions,

$$J_\lambda(\boldsymbol{x};1/a)=\sum_{\mu\overset{\mathrm{D}}{\le}\lambda}v_{\lambda\mu}(a)m_\mu(\boldsymbol{x}),\quad v_{\lambda\lambda}=1, \tag{4.2}$$

with respect to the dominance order,

$$\mu\overset{\mathrm{D}}{\le}\lambda\Leftrightarrow|\mu|=|\lambda|\text{ and }\sum_{j=1}^{l}\mu_j\le\sum_{j=1}^{l}\lambda_j\text{ for all }l=1,2,\cdots,N. \tag{4.3}$$

Since the triangular matrix $v_{\lambda\mu}(a)$ has its inverse, we have

$$m_\lambda(\boldsymbol{x}) = \sum_{\mu \overset{D}{\leq} \lambda} (v^{-1})_{\lambda\mu}(a) J_\mu(\boldsymbol{x}; 1/a). \tag{4.4}$$

Applying the above transformation to the formulae (4.1), we have

$$\begin{aligned} m_\lambda(\boldsymbol{\alpha}^{(A)\dagger}) \cdot 1 \quad &= \sum_{\mu \overset{D}{\leq} \lambda} (v^{-1})_{\lambda\mu}(a) J_\mu(\boldsymbol{\alpha}^{(A)\dagger}; 1/a) \cdot 1 \\ &= \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}^{(A)}} \sum_{\mu \overset{D}{\leq} \lambda} (v^{-1})_{\lambda\mu}(a) J_\mu(\boldsymbol{x}; 1/a) \cdot 1 \\ &= \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}^{(A)}} m_\lambda(\boldsymbol{x}) = m_\lambda(\boldsymbol{b}^{(A)+}) \cdot 1, \end{aligned} \tag{4.5a}$$

$$\begin{aligned} m_\lambda((\boldsymbol{\alpha}^{(B)\dagger})^2) \cdot 1 &= \sum_{\mu \overset{D}{\leq} \lambda} (v^{-1})_{\lambda\mu}(a) J_\mu((\boldsymbol{\alpha}^{(B)\dagger})^2; 1/a) \cdot 1 \\ &= \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}^{(B)}} \sum_{\mu \overset{D}{\leq} \lambda} (v^{-1})_{\lambda\mu}(a) J_\mu(\boldsymbol{x}^2; 1/a) \cdot 1 \\ &= \mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}^{(B)}} m_\lambda(\boldsymbol{x}^2) = m_\lambda((\boldsymbol{b}^{(B)+})^2) \cdot 1, \end{aligned} \tag{4.5b}$$

which show the "new" orthogonal basis for the A_{N-1}-Calogero model is nothing but the basis given in [16, 17], though its orthogonality and corresponding conserved operators were not given. We note that the creation-annihilation operators, b_j^+ and b_j, cannot be the same as the Dunkl operators, $\alpha_j^\dagger$ and α_j, respectively. If so, then the two sets of conserved operators $P_k(\boldsymbol{\nu})$ and I_k become the same and the corresponding simultaneous eigenfunctions also must be the same, which is contradictory. We can also directly verify it by calculating the forms of the creation-annihilation operators.

Since the transition matrix $v_{\lambda\mu}$ that relates the new and the known orthogonal bases is not a unitary but triangular matrix, it seems rather strange at first sight that the new orthogonal basis is indeed an orthogonal basis. This strange observation comes from the fact that the new sets of conserved operators $P_l(\hat{\boldsymbol{\nu}})$ are not Hermitian, but self-dual with respect to the exchange of creation-annihilation operators, $\hat{b}_l^+ \leftrightarrow \hat{b}_l$. That is the reason why the new orthogonal bases are orthogonal with respect to the inner product (3.18). On the other hand, the conserved operators given by the Dunkl operator formulation including the action to the ground state wave function, $\hat{I}_k \overset{\mathrm{def}}{=} \Psi_{\mathrm{g}} I_k \Psi_{\mathrm{g}}^{-1}$ are Hermitian operators, $\hat{I}_k^\dagger = \hat{I}_k$. That explains why the Hi-Jack and the multi-variable Laguerre polynomials are orthogonal with respect to the conventional Hermitian inner product,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} \mathrm{d}x_j |\Psi_{\mathrm{g}}^{(A)}(\boldsymbol{x})|^2 \; j_\lambda^\dagger(\boldsymbol{x}) j_\mu(\boldsymbol{x}) \propto \delta_{\lambda\mu}, \tag{4.6a}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} \mathrm{d}x_j |\Psi_{\mathrm{g}}^{(B)}(\boldsymbol{x})|^2 \; l_\lambda^\dagger(\boldsymbol{x}) l_\mu(\boldsymbol{x}) \propto \delta_{\lambda\mu}. \tag{4.6b}$$

In our normalization, the above polynomials are real functions, $j_\lambda^\dagger(\boldsymbol{x}) = j_\lambda(\boldsymbol{x})$, $l_\lambda^\dagger(\boldsymbol{x}) = l_\lambda(\boldsymbol{x})$. Comparing the two different orthogonal bases and inner products, we notice that the dual bases ${}^{(A)}\langle\lambda|$ and ${}^{(B)}\langle\lambda|$ are identified with the "rotation" of the variants of the Jack polynomial up to normalization,

$$\begin{aligned} {}^{(A)}\langle\lambda| &\propto \Psi_{\mathrm{g}}^{(A)}(\boldsymbol{x}) \sum_{\mu} v_{\lambda\mu}(a) j_\mu(\boldsymbol{x}) \\ &= \Psi_{\mathrm{g}}^{(A)}(\boldsymbol{x}) \sum_{\mu\rho} v_{\lambda\mu}(a) v_{\mu\rho}(a) M_\rho(\boldsymbol{x}), \end{aligned} \tag{4.7a}$$

$$\begin{aligned} {}^{(B)}\langle\lambda| &\propto \Psi_{\mathrm{g}}^{(B)}(\boldsymbol{x}) \sum_{\mu} v_{\lambda\mu}(a) l_\mu(\boldsymbol{x}) \\ &= \Psi_{\mathrm{g}}^{(B)}(\boldsymbol{x}) \sum_{\mu\rho} v_{\lambda\mu}(a) v_{\mu\rho}(a) Y_\rho(\boldsymbol{x}). \end{aligned} \tag{4.7b}$$

The above identification of the dual bases is, at least, valid in the consideration of the inner product. Thus both new and known orthogonal bases give the same thermodynamics quantities calculated by the trace formula. However, we should note that a naive identification of the dual bases as functions themselves has difficulties. Further considerations on this point are left for future studies.

From the discussions above, we conclude that each Calogero model has at least two sets of commuting conserved operators which are algebraically inequivalent to each other. We also conclude that two conserved operators respectively picked up from two different sets do not commute $[P_n(\hat{\boldsymbol{\nu}}), \hat{I}_k] \neq 0$, for $n, k \neq 1$. Its Hilbert space also has two different orthogonal bases with respect to two "different" inner products that respectively correspond to the simultaneous eigenfunctions of two sets of commuting conserved operators. This peculiar fact may be due to the large degeneracy of the eigenvalue of the Calogero models,

$$H_{\mathrm{C}}^{(A)} M_\lambda(\boldsymbol{x}; 1/a, \omega) = \omega|\lambda| M_\lambda(\boldsymbol{x}; 1/a, \omega), \tag{4.8a}$$

$$H_{\mathrm{C}}^{(B)} Y_\lambda(\boldsymbol{x}; 1/a, 1/b, \omega) = 2\omega|\lambda| Y_\lambda(\boldsymbol{x}; 1/a, 1/b, \omega). \tag{4.8b}$$

For a particular eigenvalue, say ωn and $2\omega n$, the degeneracy is given by the number of the Young diagrams λ such that $|\lambda| = n$, namely, the number of partitions not exceeding N parts.

The above peculiar story reminds us of another example of such a story, i.e., the wave functions of the hydrogen atom [36]. The well-known eigenfunctions for the hydrogen atom,

$$H_{\mathrm{H}} \Phi(\boldsymbol{r}) \stackrel{\mathrm{def}}{=} \Big(\frac{1}{2m}\boldsymbol{p}^2 - \frac{k}{r}\Big)\Phi(\boldsymbol{r}) = E\Phi(\boldsymbol{r}), \tag{4.9}$$

where $\boldsymbol{p}$, $\boldsymbol{r}$ and r denote the momentum operator, the coordinate vector and its norm in three dimensional space, are obtained by separation of variables in spherical coordinates. They are the simultaneous eigenfunctions of the Hamiltonian H_{H}, the total angular

momentum $\boldsymbol{L}^2$ and its z-axis component L_z where $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, and are expressed by the product of the spherical harmonics and the associated Laguerre polynomial. On the other hand, the Schrödinger equation can be solved by separation of variables in parabolic coordinates and results in the wave functions that contain a product of two associated Laguerre polynomials. They simultaneously diagonalize H_{H}, L_z and the z-axis component of the Runge-Lenz-Pauli vector M_z,

$$\boldsymbol{M} \stackrel{\text{def}}{=} \frac{1}{2m}(\boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{p}) - \frac{k}{r}\boldsymbol{r}. \tag{4.10}$$

This is another conserved operator of the hydrogen atom. The total angular momentum and the Runge-Lenz-Pauli vector are algebraically different because of the relation, $[M_3, \boldsymbol{L}^2] \neq 0$. The two solutions respectively derived by two ways are different, and the former is a linear combination of the latter and vice versa. This is quite similar to what we have observed for the Calogero models. Behind the story for the hydrogen atom, there is the $O(4)$ dynamical symmetry. We stress that the quest for some hidden dynamical symmetry for the Calogero model must be an interesting future problem.

A similar transformation for the Sutherland model into a decoupled system has not been found so far. This seems rather strange at first because the commuting conserved operators and the Dunkl operators of the Calogero and Sutherland models share the same algebraic structure, and become exactly the same in the limit, $\omega \to \infty$ [21, 22, 28]. The difference of the two models is the structure of the Hamiltonian. While the Calogero Hamiltonian is the simplest conserved operator I_1, the Sutherland Hamiltonian corresponds to the second conserved operator I_2. We have proved that the second conserved operator I_2 can not be constructed from the power sums of the number operators $P_n(\boldsymbol{\nu})$. We think that the point causes the critical difficulty in the application of such a similarity transformation method to the Sutherland model.

5. Summary

We have studied an algebraic construction of new orthogonal bases for the A_{N-1}- and B_N-Calogero models and their conserved operators by means of similarity transformations to decoupled quantum harmonic oscillators. Our idea is just pulling the number operators and number states back to the Hilbert space of the Calogero models by the similarity transformations. We have pointed out that a delicate property of the exponentiation of the Lassalle operators which yields essential singularities requires its operands to be symmetric functions for both models. For the case of the B_N-Calogero model, the property further demands its operands to be even functions. Thus we introduce the symmetrized number state, which is nothing but the bosonic state for the quantum harmonic oscillators, and restrict its parity to be even for the B_N case. The symmetrized number state is uniquely determined as the symmetric simultaneous

eigenfunctions for the power sums of the number operators, $P_l(\boldsymbol{n})$, $l = 1, 2, \cdots, N$, and spans the orthogonal basis of the N decoupled quantum harmonic oscillators. Since the conserved operators $P_l(\hat{\boldsymbol{\nu}})$, $l = 2, \cdots, N$, are not Hermitian, the definition of the inner products for the new orthogonal bases is different from the conventional Hermitian inner product. Consequently we have obtained two sets of commuting conserved operators and two sets of orthogonal symmetric polynomials as the simultaneous eigenfunctions of the conserved operators. The orthogonal symmetric polynomials span new orthogonal bases for the A_{N-1}- and B_N-Calogero models.

The two Calogero models have known orthogonal bases that are spanned by the Hi-Jack and the multivariable Laguerre polynomials, which are uniquely identified as the simultaneous eigenfunctions of the conserved operators constructed by the Dunkl operator formulations for the models. Comparison of the new and the known orthogonal bases reveals that they are different, though both are considered to be multivariable generalizations of the Hermite polynomials and those of the Laguerre polynomials. The fact means that the conserved operators given by the similarity transformations and those given by the Dunkl operator formulations do not commute. Thus the A_{N-1}- and B_N-Calogero models respectively have two sets of commuting conserved operators that are algebraically inequivalent to each other and two orthogonal bases that respectively correspond to the simultaneous eigenfunctions for each set of commuting conserved operators. We have conjectured that this peculiar fact implies some hidden dynamical symmetry for the models, as is the case with the hydrogen atom. We have shown triangularity and integrality that appear in the expansion form of the new orthogonal symmetric polynomials with respect to the monomial symmetric functions. For the A_{N-1} case, the "new" orthogonal basis turns out to be the same basis that was given by Brink, Hansson, Konstein and Vasiliev. Still we stress it is new as an orthogonal basis because its orthogonality and corresponding conserved operators are presented in this paper. We have discussed the difficulty in the application of the similarity transformation method to the Sutherland model from the algebraic structural point of view.

In short, we have completed a construction of the symmetric Fock space of the Calogero models. Recently, an extension of the similarity transformation method to the non-symmetric case, which gives complete Fock space for the Calogero Hamiltonians, was announced [37]. We expect that these Fock spaces will be a useful tool for calculation of various kinds of quantities such as the Green function and correlation functions of the Calogero models.

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Appendix A. Dunkl operator formulation

We briefly summarize the Dunkl operator formulations for the A_{N-1}- and B_N-Calogero and Sutherland models, which are essential tools to study the algebraic structure, integrability and variants of the Jack polynomial [27, 28].

Appendix A.1. A_{N-1}-Calogero and Sutherland models

The Dunkl operators for the A_{N-1}-Sutherland model are given by

$$\begin{aligned}
&\nabla_l^{(A)} = \frac{\partial}{\partial x_l} + a\sum_{\substack{k=1\\k\neq l}}^{N} \frac{1}{x_l - x_k}(1 - K_{lk}),\\
&x_l,\\
&D_l^{(A)} = x_l \nabla_l^{(A)},
\end{aligned} \tag{A1}$$

where K_{lk} is the coordinate exchange operator that is defined by the action on multi-variable functions of $\boldsymbol{x} = (x_1, \cdots, x_N)$,

$$(K_{lk} f)(\cdots, x_l, \cdots, x_k, \cdots) = f(\cdots, x_k, \cdots, x_l, \cdots). \tag{A2}$$

Commutation relations among the Dunkl operators are given by

$$\begin{aligned}
&[\nabla_l^{(A)}, \nabla_m^{(A)}] = 0, \qquad [x_l, x_m] = 0,\\
&[\nabla_l^{(A)}, x_m] = \delta_{lm}(1 + a\sum_{\substack{k=1\\k\neq l}}^{N} K_{lk}) - a(1 - \delta_{lm})K_{lm},\\
&[D_l^{(A)}, D_m^{(A)}] = a(D_m^{(A)} - D_l^{(A)})K_{lm},\\
&\nabla_l^{(A)} \cdot 1 = 0.
\end{aligned} \tag{A3}$$

A set of mutually commuting conserved operators for the A_{N-1}-Sutherland model are given by

$$L_k^{(A)} = \sum_{l=1}^{N} (D_l^{(A)})^k \Big|_{\mathrm{Sym}}, \quad [L_k^{(A)}, L_m^{(A)}] = 0 \quad k, m = 1, 2, \cdots, N, \tag{A4}$$

where the symbol $\Big|_{\mathrm{Sym}}$ means that the action of the operator is restricted to symmetric functions, $K_{lk} f(\boldsymbol{x}) = f(\boldsymbol{x})$. The Hamiltonian of the A_{N-1}-Sutherland model corresponds to the second conserved operator $L_2^{(A)}$. In terms of the A_{N-1}-Sutherland Hamiltonian, the Jack symmetric polynomials $J_\lambda(\boldsymbol{x}; 1/a)$ are uniquely defined by [4]

$$\begin{aligned}
&L_2^{(A)} J_\lambda(\boldsymbol{x}; 1/a) = \sum_{k=1}^{N} (\lambda_k^2 + a(N + 1 - 2k)\lambda_k) J_\lambda(\boldsymbol{x}; 1/a),\\
&J_\lambda(\boldsymbol{x}; 1/a) = \sum_{\mu \overset{\mathrm{D}}{\leq} \lambda} v_{\lambda\mu}(a) m_\mu(\boldsymbol{x}), \quad v_{\lambda\lambda}(a) = 1,
\end{aligned} \tag{A5}$$

where λ and μ are the Young diagrams. The symbol $\overset{\mathrm{D}}{\leq}$ is the dominance order among the Young diagrams [4],

$$\mu\overset{\mathrm{D}}{\leq}\lambda \Leftrightarrow \sum_{k=1}^{N}\mu_k=\sum_{k=1}^{N}\lambda_k \text{ and } \sum_{k=1}^{l}\mu_k\leq\sum_{k=1}^{l}\lambda_k \text{ for all } l. \tag{A6}$$

Note that the dominance order is not a total order but a partial order. Since the Jack symmetric polynomials diagonalize all the mutually commuting conserved operators $L_k^{(A)}$ simultaneously, they form the orthogonal basis of the A_{N-1}-Sutherland model [1–11].

A similar formulation is also applicable to the A_{N-1}-Calogero model. The Dunkl operators for the A_{N-1}-Calogero model are

$$\begin{aligned}
&\alpha_l^{(A)}=\nabla_l^{(A)},\\
&\alpha_l^{(A)\dagger}=-\frac{1}{2\omega}\nabla_l^{(A)}+x_l,\\
&d_l^{(A)}=\alpha_l^{(A)\dagger}\alpha_l^{(A)}.
\end{aligned} \tag{A7}$$

The above Dunkl operators are a one-parameter deformation of those for the A_{N-1}-Sutherland model and the former reduces to the latter in the limit, $\omega\to\infty$. The Dunkl operators for the A_{N-1}-Calogero model satisfy the commutation relations,

$$\begin{aligned}
&[\alpha_l^{(A)},\alpha_m^{(A)}]=0, \qquad [\alpha_l^{(A)\dagger},\alpha_m^{(A)\dagger}]=0,\\
&[\alpha_l^{(A)},\alpha_m^{(A)\dagger}]=\delta_{lm}(1+a\sum_{\substack{k=1\\k\neq l}}^{N}K_{lk})-a(1-\delta_{lm})K_{lm},\\
&[d_l^{(A)},d_m^{(A)}]=a(d_m^{(A)}-d_l^{(A)})K_{lm},\\
&\alpha_l^{(A)}\cdot 1=0,
\end{aligned} \tag{A8}$$

which are exactly the same as those for the Sutherland model (A3). Thus the Dunkl operators for the A_{N-1}-Calogero and Sutherland models share the same algebraic structure [21, 22, 28]. Commuting conserved operators for the Calogero model are obtained in a similar way to (A4),

$$I_k^{(A)}=\sum_{l=1}^{N}(d_l^{(A)})^k\Big|_{\mathrm{Sym}}, \quad [I_k^{(A)},I_m^{(A)}]=0 \quad k,m=1,2,\cdots,N. \tag{A9}$$

The correspondences between the operators for the two models are

$$\alpha_l^{(A)}\leftrightarrow\nabla_l^{(A)}, \quad \alpha_l^{(A)\dagger}\leftrightarrow x_l, \quad d_l^{(A)}\leftrightarrow D_l^{(A)}, \quad I_l^{(A)}\leftrightarrow L_l^{(A)}. \tag{A10}$$

The commutator algebra of these operators is translated by the correspondences in (A10). The first conserved operator $I_1^{(A)}$ is identified with the Hamiltonian of the A_{N-1}-Calogero model, $\omega I_1^{(A)}=H_{\mathrm{C}}^{(A)}$. Because of these correspondences, the Jack symmetric polynomials are transformed into the orthogonal basis for the A_{N-1}-Calogero model which are called Hi-Jack symmetric polynomials, $j_\lambda(\boldsymbol{x})=J_\lambda(\alpha_1^{(A)\dagger},\cdots,\alpha_N^{(A)\dagger})\cdot 1$ [22, 23].

Appendix A.2. B_N-Calogero and Sutherland models

Similar to the A_{N-1} case, there are Dunkl operator formulations for the B_N-Calogero and Sutherland models. The Dunkl operators for the B_N-Sutherland model are

$$\begin{aligned}
\nabla_l^{(B)} &= \frac{\partial}{\partial x_l} + \frac{b}{x_l}(1 - P_l) \\
&\quad + a\sum_{\substack{k=1 \\ k\neq l}}^{N}\Big(\frac{1}{x_l - x_k}(1 - K_{lk}) + \frac{1}{x_l + x_k}(1 - P_l P_k K_{lk})\Big), \\
x_l&, \\
D_l^{(B)} &= x_l \nabla_l^{(B)},
\end{aligned} \tag{A11}$$

where K_{lk} and P_l are elements of the B_N-type Weyl group, K_{lk} is the coordinate exchange operator whose action is same as in the A_{N-1} case and P_l is the reflection operator whose action on multivariable functions is defined by

$$(P_l f)(\cdots, x_l, \cdots) = f(\cdots, -x_l, \cdots). \tag{A12}$$

Commutation relations among the operators are given by

$$\begin{aligned}
&[\nabla_l^{(B)}, \nabla_m^{(B)}] = 0, \qquad [x_l, x_m] = 0, \\
&[\nabla_l^{(B)}, x_m] = \delta_{lm}(1 + a\sum_{\substack{k=1 \\ k\neq m}}^{N}(1 + P_m P_k)K_{mk} + 2bP_m) \\
&\qquad\qquad\qquad -a(1 - \delta_{lm})(1 - P_l P_m)K_{lm}, \\
&[D_l^{(B)}, D_m^{(B)}] = a(D_m^{(B)} - D_l^{(B)})(1 + P_l P_m)K_{lk}, \\
&\nabla_l^{(B)} \cdot 1 = 0,
\end{aligned} \tag{A13}$$

and the commuting conserved operators are

$$L_k^{(B)} = \sum_{l=1}^{N}(D_l^{(B)})^k\Big|_{\mathrm{Sym,Even}}, \quad [L_k^{(B)}, L_m^{(B)}] = 0, \quad k, m = 1, 2, \cdots, N, \tag{A14}$$

where the symbol $\Big|_{\mathrm{Sym,Even}}$ denotes the restriction of the operand to symmetric functions with even parity. We note that the restriction of the operands of the Dunkl operators $D_l^{(B)}$ to even functions $\Big|_{\mathrm{Even}}$ yields

$$D_l^{(B)}\Big|_{\mathrm{Even}} = x_l\frac{\partial}{\partial x_l} + a\sum_{\substack{k=1 \\ k\neq l}}^{N}\frac{2x_l^2}{x_l^2 - x_k^2}(1 - K_{lk}). \tag{A15}$$

Comparing (A15) with (A1), we notice that $D_l^{(B)}\Big|_{\mathrm{Even}}$ is equivalent to $2D_l^{(A)}$ with the change of variables, $x_l \to x_l^2/2$. As a consequence, the symmetric simultaneous eigenfunctions of the conserved operators $L_k^{(B)}$ with even parity are

given by the Jack symmetric polynomials whose arguments x_l are replaced with $x_l^2/2$, $J_\lambda(x_1^2/2,\cdots,x_N^2/2)=2^{-|\lambda|}J_\lambda(\boldsymbol{x}^2)$. They form the orthogonal basis of the B_N-Sutherland model.

The Dunkl operators for the B_N-Calogero model are

$$\begin{aligned}
&\alpha_l^{(B)}=\nabla_l^{(B)},\\
&\alpha_l^{(B)\dagger}=-\frac{1}{2\omega}\nabla_l^{(B)}+x_l,\\
&d_l^{(B)}=\alpha_l^{(B)\dagger}\alpha_l^{(B)}.
\end{aligned}\tag{A16}$$

Commutation relations among these operators are given by

$$\begin{aligned}
&[\alpha_l^{(B)},\alpha_m^{(B)}]=0,\qquad [\alpha_l^{(B)\dagger},\alpha_m^{(B)\dagger}]=0,\\
&[\alpha_l^{(B)},\alpha_m^{(B)\dagger}]=\delta_{lm}(1+a\sum_{\substack{k=1\\k\neq m}}^{N}(1+P_mP_k)K_{mk}+2bP_m)\\
&\qquad\qquad -a(1-\delta_{lm})(1-P_lP_m)K_{lm},\\
&[d_l^{(B)},d_m^{(B)}]=a(d_m^{(B)}-d_l^{(B)})(1+P_lP_m)K_{lm},\\
&\alpha_l^{(B)}\cdot 1=0,
\end{aligned}\tag{A17}$$

and the mutually commuting conserved operators are

$$I_k^{(B)}=\sum_{l=1}^{N}(d_l^{(B)})^k\Big|_{\text{Sym,Even}},\quad [I_k^{(B)},I_m^{(B)}]=0,\quad k,m=1,2,\cdots,N.\tag{A18}$$

In a similar way to the translation between the A_{N-1}-Calogero and Sutherland models, the simultaneous eigenfunctions of the above conserved operators are obtained by putting $(\alpha_i^{(B)\dagger})^2/2$ into the arguments of the Jack polynomials, $2^{-|\lambda|}l_\lambda(\boldsymbol{x})=J_\lambda((\alpha_1^\dagger)^2/2,\cdots,(\alpha_N^\dagger)^2/2)\cdot 1=2^{-|\lambda|}J_\lambda((\boldsymbol{\alpha}^\dagger)^2)\cdot 1$, which form the orthogonal basis of the B_N-Calogero model.

Appendix B. Cancellation of essential singularities

We show how the transformation (2.17) of the non-symmetric number states causes the essential singularity and how we can escape from it. We note that the transformation of the number state is rewritten as

$$T|n_1,\cdots,n_N\rangle=\mathrm{e}^{-\frac{1}{4\omega}\mathcal{O}_{\mathrm{L}}}x_1^{n_1}x_2^{n_2}\cdots x_N^{n_N}.\tag{B1}$$

First, we consider the A_{N-1}-Calogero model. Action of Lassalle operator $\mathcal{O}_{\mathrm{L}}^{(A)}$ on a monomial $x_1^{n_1}x_2^{n_2}\cdots x_N^{n_N}$ yields

$$\mathcal{O}_{\mathrm{L}}^{(A)}x_1^{n_1}x_2^{n_2}\cdots x_N^{n_N}=\sum_{l=1}^{N}n_l(n_l-1)x_1^{n_1}\cdots x_l^{n_l-2}\cdots x_N^{n_N}$$

$$+ a \sum_{\substack{l,m=1 \\ l\neq m}}^{N} \frac{1}{x_l - x_m}(n_l x_m - n_m x_l) x_1^{n_1} \cdots x_l^{n_l - 1} \cdots x_m^{n_m - 1} \cdots x_N^{n_N} . \tag{B2}$$

As we see, action of the Lassalle operator generally generates poles at $x_l = x_m$ in the second term. The action of the exponentiation of the Lassalle operator means the action of the Lassalle operator infinitely many times, which develops such poles into essential singularities. We can remove such poles by symmetrization. Acting the Lassalle operator $\mathcal{O}_{\mathrm{L}}^{(A)}$ on a symmetrized monomial $m_\lambda(\boldsymbol{x})$ (3.5), we have

$$\begin{aligned}
\mathcal{O}_{\mathrm{L}}^{(A)} m_\lambda &= a \sum_{\substack{l,m=1 \\ l\neq m}}^{N} \sum_{\substack{\sigma \in S_N \\ \text{distinct}}} \frac{\lambda_{\sigma(l)} x_l^{\lambda_{\sigma(l)}-1} x_m^{\lambda_{\sigma(m)}} - \lambda_{\sigma(m)} x_l^{\lambda_{\sigma(l)}} x_m^{\lambda_{\sigma(m)}-1}}{x_l - x_m} x_1^{\lambda_{\sigma(1)}} \cdots \overset{l}{\vee} \cdots \overset{m}{\vee} \cdots x_N^{\lambda_{\sigma(N)}} \\
&\qquad + (\text{some symmetrized monomials}) \\
&= a \sum_{\substack{l,m=1 \\ l\neq m}}^{N} \sum_{\substack{\sigma \in S_N \\ \text{distinct}}} \lambda_{\sigma(l)} \frac{x_l^{\lambda_{\sigma(l)}-1} x_m^{\lambda_{\sigma(m)}} - x_m^{\lambda_{\sigma(l)}-1} x_l^{\lambda_{\sigma(m)}}}{x_l - x_m} x_1^{\lambda_{\sigma(1)}} \cdots \overset{l}{\vee} \cdots \overset{m}{\vee} \cdots x_N^{\lambda_{\sigma(N)}} \\
&\qquad + (\text{some symmetrized monomials}) \\
&= (\text{some symmetrized monomials}),
\end{aligned} \tag{B3}$$

where $\overset{l}{\vee}$ denotes a missing x_l. We have used the fact that the numerator in the second expression has a factor $(x_l - x_m)$, which cancels the denominator out. Thus we have removed poles by symmetrization.

In a similar way, essential singularities appear in the B_N case. We act the B_N-Lassalle operator $\mathcal{O}_{\mathrm{L}}^{(B)}$ on a monomial $x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}$ and get

$$\begin{aligned}
\mathcal{O}_{\mathrm{L}}^{(B)} x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} &= \sum_{l=1}^{N} (n_l(n_l - 1) + 2b n_l) x_1^{n_1} \cdots x_l^{n_l - 2} \cdots x_N^{n_N} \\
&\quad + 2a \sum_{\substack{l,m=1 \\ l\neq m}}^{N} \frac{n_l - n_m}{x_l^2 - x_m^2} x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} .
\end{aligned} \tag{B4}$$

The second term yields poles at $x_l = x_m$ and $x_l = -x_m$. In addition to those, the first term also yields a pole at $x_l = 0$ when $n_l = 1$. Considering successive actions of the Lassalle operator, we conclude that the poles of the second type appear when the powers n_l are odd. Thus we have to restrict the operand to even functions. Symmetrizing (B4), we act the Lassalle operator on an even symmetrized monomial,

$$\begin{aligned}
\mathcal{O}_{\mathrm{L}}^{(B)} m_\lambda &= 2a \sum_{\substack{l,m=1 \\ l\neq m}}^{N} \sum_{\substack{\sigma \in S_N \\ \text{distinct}}} \frac{\lambda_{\sigma(l)} - \lambda_{\sigma(m)}}{x_l^2 - x_m^2} x_1^{\lambda_{\sigma(1)}} \cdots x_N^{\lambda_{\sigma(N)}} \\
&\qquad + (\text{some even symmetrized monomials})
\end{aligned}$$

$$= 2a \sum_{\substack{l,m=1 \\ l \neq m}}^{N} \sum_{\substack{\sigma \in S_N \\ \text{distinct}}} \lambda_{\sigma(l)} \frac{x_l^{\lambda_{\sigma(l)}} x_m^{\lambda_{\sigma(m)}} - x_m^{\lambda_{\sigma(l)}} x_l^{\lambda_{\sigma(m)}}}{x_l^2 - x_m^2} x_1^{\lambda_{\sigma(1)}} \cdots \overset{l}{\vee} \cdots \overset{m}{\vee} \cdots x_N^{\lambda_{\sigma(N)}}$$
$$+(\text{some even symmetrized monomials})$$
$$= (\text{some even symmetrized monomials}). \tag{B5}$$

We have used the fact that the numerator in the second expression has a factor $(x_l^2 - x_m^2)$ when $\lambda_{\sigma(l)}$ and $\lambda_{\sigma(m)}$ are even. Consequently, we can escape from the essential singularity by restricting the operands of the B_N-Lassalle operator to symmetric function with even parity.

Appendix C. Explicit forms of new orthogonal bases

We show explicit forms of some of the new orthogonal bases and variants of the Jack polynomial. They are expressed in terms of the monomial symmetric functions.

The first seven of the new orthogonal symmetric polynomials for the A_{N-1}-Calogero model are

$$\begin{aligned}
&M_0 = m_0 = 1, \quad M_1 = m_1, \\
&M_2 = m_2 - \frac{1}{2\omega} N[(N-1)a + 1] m_0, \\
&M_{1^2} = m_{1^2} + \frac{1}{4\omega} N(N-1)a\, m_0, \\
&M_3 = m_3 - \frac{3}{2\omega}[(N-1)a + 1] m_1, \\
&M_{2,1} = m_{2,1} - \frac{1}{2\omega}(N-1)[(N-3)a + 1] m_1, \\
&M_{1^3} = m_{1^3} + \frac{1}{4\omega}(N-1)(N-2)a\, m_1,
\end{aligned} \tag{C1}$$

and the first four of those for the B_N-Calogero model are

$$\begin{aligned}
&Y_0 = m_0 = 1, \\
&Y_1 = m_1 - \frac{1}{2\omega} N[2(N-1)a + 2b + 1] m_0, \\
&Y_2 = m_2 - \frac{1}{\omega}[4(N-1)a + 2b + 3] m_1 \\
&\qquad + \frac{1}{4\omega^2} N[4(N-1)a + 2b + 3][2(N-1)a + 2b + 1] m_0, \\
&Y_{1^2} = m_{1^2} - \frac{1}{2\omega}(N-1)[2(N-2)a + 2b + 1] m_1 \\
&\qquad + \frac{1}{8\omega^2} N(N-1)[2(N-2)a + 2b + 1][2(N-1)a + 2b + 1] m_0.
\end{aligned} \tag{C2}$$

We also present the first seven of the Hi-Jack polynomials,

$$
\begin{aligned}
j_0 &= M_0 = 1, \quad j_1 = M_1 = m_1, \\
j_2 &= M_2 + \frac{2a}{a+1} M_{1^2} \\
&= m_2 + \frac{2a}{a+1} m_{1^2} - \frac{1}{2\omega} \frac{N(Na+1)}{a+1} m_0, \\
j_{1^2} &= M_{1^2}, \\
j_3 &= M_3 + \frac{3a}{a+2} M_{2,1} + \frac{6a^2}{(a+1)(a+2)} M_{1^3} \\
&= m_3 + \frac{3a}{a+2} m_{2,1} + \frac{6a^2}{(a+1)(a+2)} m_{1^3} \\
&\quad - \frac{3}{2\omega} \frac{(Na+1)(Na+2)}{(a+1)(a+2)} m_1, \\
j_{2,1} &= M_{2,1} + \frac{6a}{2a+1} M_{1^3} \\
&= m_{2,1} + \frac{6a}{2a+1} m_{1^3} + \frac{1}{2\omega} \frac{(Na-1)(Na+1)(a-1)}{2a+1} m_1, \\
j_{1^3} &= M_{1^3},
\end{aligned} \tag{C3}
$$

and the first four of the multivariable Laguerre polynomials,

$$
\begin{aligned}
l_0 &= Y_0 = 1, \quad l_1 = Y_1, \\
l_2 &= Y_2 + \frac{2a}{a+1} Y_{1^2} \\
&= m_2 + \frac{2a}{a+1} m_{1^2} \\
&\quad - \frac{1}{\omega} \frac{1}{a+1} [(N-1)(2Na+2b+5)a + (a+1)(2b+3)] m_1 \\
&\quad + \frac{1}{4\omega^2} \frac{1}{a+1} N[2(N-1)a+2b+1] \\
&\quad \times [(N-1)(2Na+2b+5)a + (a+1)(2b+3)] m_0, \\
l_{1^2} &= Y_{1^2}.
\end{aligned} \tag{C4}
$$

References

[1] Sutherland B 1971 *Phys. Rev.* A **4** 2019

[2] Sutherland B 1972 *Phys. Rev.* A **5** 1372

[3] Jack H 1970 *Proc. R. Soc. Edinburgh* Sect. A **69** 1

[4] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon Press) 2nd ed.

[5] Stanley R P 1989 *Adv. Math.* **77** 76

[6] Forrester P J 1992 *Nucl. Phys.* B **388** 671

[7] Forrester P J 1994 *Nucl. Phys.* B **416** 377

[8] Ha Z N C 1994 *Phys. Rev. Lett.* **73** 1574

[9] Ha Z N C 1995 *Nucl. Phys.* B **435** 604

[10] Lapointe L and Vinet L 1995 *IMRN* **9** 419

[11] Lapointe L and Vinet L 1996 *Commun. Math. Phys.* **178** 425

[12] Calogero F 1971 *J. Math. Phys.* **12** 419

[13] Sutherland B 1971 *J. Math. Phys.* **12** 246

[14] Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313

[15] Perelomov A M 1971 *Theor. Math. Phys.* **6** 263

[16] Brink L, Hansson T H and Vasiliev M A 1992 *Phys. Lett.* B **286** 109

[17] Brink L, Hansson T H, Konstein S and Vasiliev M A 1993 *Nucl. Phys.* B **401** 591

[18] Ujino H and Wadati M 1995 *Chaos, Solitons & Fractals* **5** 109

[19] Ujino H and Wadati M 1994 *J. Phys. Soc. Japan* **63** 3585

[20] Ujino H and Wadati M 1995 *J. Phys. Soc. Japan* **64** 2703

[21] Ujino H and Wadati M 1996 *J. Phys. Soc. Japan* **65** 653

[22] Ujino H and Wadati M 1996 *J. Phys. Soc. Japan* **65** 2423

[23] Ujino H and Wadati M 1997 *J. Phys. Soc. Japan* **66** 345

[24] Baker T H and Forrester P J 1997 *Commun. Math. Phys.* **188** 175

[25] van Diejen J F 1997 *Commun. Math. Phys.* **188** 467

[26] Kakei S 1997 *J. Phys. A: Math. Gen.* **30** L535

[27] Dunkl C F 1989 *Trans. Amer. Math. Soc.* **311** 167

[28] Polychronakos A P 1992 *Phys. Rev. Lett.* **69** 703

[29] Yamamoto T 1995 *Phys. Lett.* A **208** 293

[30] Gurappa N and Panigrahi P K 1997 Equivalence of Calogero-Sutherland model to free harmonic oscillator *preprint* cond-mat/9710035

[31] Gurappa N and Panigrahi P K 1997 Mapping of the B_N-type Calogero-Sutherland-Moser system to decoupled harmonic oscillator *preprint* quant-ph/9710019

[32] Sogo K 1996 *J. Phys. Soc. Japan* **65** 3097

[33] Ujino H and Wadati M 1998 *J. Phys. Soc. Japan* **67** 1

[34] Lassalle M 1991 *C. R. Acad. Sci. Paris t. Séries I* **312** 725

[35] Lassalle M 1991 *C. R. Acad. Sci. Paris t. Séries I* **313** 579

[36] Schiff L I 1968 *Quantum Mechanics* (New York: McGrawhill Book Company) 3rd ed.

[37] Ujino H, Nishino A and Wadati M 1997 New non-symmetric orthogonal basis for the Calogero model with distinguishable particles *preprint* UT/Wu019